MTH 406: Differential geometry of curves and surfaces

Homework V: Tangent Space

Problems for practice

1. Consider the torus T^2 given by parametrization in Lesson Plan 2.1(ii)(b). Describe the tangent plane at an ordinary point of the $u = 0$ parameter curve and at an arbitrary point of the $v = 0$ parameter curve of T^2.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map, and then let S be a regular surface that is left invariant by L. Then $L|_S$ is a differentiable map, and

$$dL_p(v) = L(v), \forall p \in S \text{ and } v \in T_p(S).$$

3. If the coordinate neighborhood of a regular surface can be parametrized in the form

$$f(u, v) = \alpha_1(u) + \alpha_2(v),$$

where the α_i are parametrized curves, show that the tangent plane along a fixed coordinate curve of this neighborhood are all parallel to a line.

4. Show that the tangent planes of a surface given by $z = xf(y/x)$, $x \neq 0$, where f is a differentiable function, all pass through the origin.

5. Let S be a regular surface, $p \in S$, and let N be a unit vector of $T_p(S)$. show that if (p_n) is sequence of points such that $p_n \to p$, then

$$\lim_{n\to\infty} \left\langle N, \frac{p_n - p}{\|p_n - p\|} \right\rangle = 0.$$

6. Let G_f denote the graph of a function f. Let $U(\subset \mathbb{R}^2) \to \mathbb{R}$ be a smooth map, where U is open. If $p = (x, y) \in U$, and $q = (x, y, f(x, y)) \in G_f$, show that

$$T_q(G_f) = G_{df_q}.$$

7. Let $S \subset \mathbb{R}^3$ be a regular surface, and let $q \in \mathbb{R}^3 \setminus S$. Show that if p is the global minimum of the function

$$f : S \to \mathbb{R} : p \overset{f}{\mapsto} \|p - q\|,$$

then $\langle q - p, v\rangle = 0$, for all $v \in T_p(S)$.

8. Prove that if a regular surface S meets a plane P at a single point $p \in S$, then $T_p(S) = P$.

9. Show that is all normals to a connected regular surface pass through a fixed point, then the surface is contained in a sphere.

10. Show that if all normals to a connected regular surface meet a fixed line, then S is a piece of a surface of revolution.